Limit Evaluation Techniques – Substitution

Limits to a number

Since we are talking about substitution, realize that horizontal asymptotes are not part of this discussion. We are considering what happens to the value of a function as the *x*-value is approaching a number we can point to on a number line. Limits "to infinity" come later.

Continuous Functions

As we learned, continuous functions are **defined** as functions where the limit *equals* the function value. What you should realize is that every function you've ever studied is continuous (or mostly so).

Limits of continuous functions

To evaluate the limit of a continuous function, just substitute the approached *x*-value into the function. In symbolic notation:

$$\lim_{x \to a} f(x) = f(a).$$

This works every time f is continuous at $x=a$.

Limit can be zero or infinity

The zero part should be obvious. For example, $\lim_{x \to 0} x = 0$. The more difficult part is that – when you substitute – you ***never*** get an answer of infinity. Despite the sad truth that millions of calculus students over the years have internalized the following "fact": $\frac{1}{0} = \infty$.

IT DOES NOT WORK THAT WAY. Infinity is not a number. It also leads to the horrible fallacy that you can multiply infinity by 0 and get a number. UGH.

The example of $\frac{1}{x}$

The most common way of getting a limit **of** infinity is to have something like: $\lim_{x \to 0+} \frac{1}{x}$.

But what is happening is that we are substituting 0 for *x*, and we encounter a forbidden situation: division by zero. This indicates that the **function** we are considering ***is not continuous*** at that value of *x*! We did the same thing in PreCalc to uncover vertical asymptotes.

If the function is not continuous, then we can't use substitution to evaluate the limit. We'll consider this case (of an infinite limit) at another time.

The limit can fail to exist

Please understand that we are now talking about functions you would have formerly considered weird. Calculus involves the study of many different functions in order to KNOW. We learn what works, what does not work, and why. Here's one you might never have looked at before …

Consider $f(x) = \lfloor x \rfloor$. This function is called the 'greatest integer function' or 'floor' function. It represents the largest integer that is less than *x*. Some examples: $\lfloor 3 \rfloor = 3; \lfloor 3.98 \rfloor = 3; \lfloor -3 \rfloor = -3; \lfloor -3.45 \rfloor = -4$. You can graph this using the int function (Math => Num => int). Use the standard window to see what the graph looks like (it's called a **step** function). For obvious reasons, the limit of the floor function exists for every real number except integers. (The obvious reasons are that the left-hand limit approaches one integer; the right-hand limit approaches a different integer.).

Now – you are able to substitute integers into this function, but that **does not give you the limit** because the function is not continuous at integers! So – when faced with a non-obvious function, you need to understand how the function behaves around that approached-to point to know if substitution is a valid method.

Some closure

So – when encountering a limit, it will often help to try substitution first. If the substituting reveals a problem, then you must consider another method. If you get a value, AND if the function is 'well behaved' (continuous), then you have your answer. What functions are continuous?

Any polynomial; any rational function (where it's defined); any trigonometric function (where it's defined); and exponential function; any logarithmic function; any inverse trigonometric function (where it's defined).; any differentiable function; and any power function (where it's defined).

Practice

It's kind of boring to create (or work through) worksheets that practice only this. Here are a few problems though. Remember that the correct answer will be either the limit OR a determination from the substitution attempt that you have to use a different approach.

1. $\lim\limits_{x\to 0}\dfrac{6x-9}{x^3-12x+3}$ 2. $\lim\limits_{x\to 3+}\dfrac{x}{x-3}$ 3. $\lim\limits_{x\to 3}\dfrac{x^2-2x}{x+1}$ 4. $\lim\limits_{y\to 1-}\dfrac{(y-1)(y-2)}{y+1}$

5, $\lim\limits_{y\to 5}\dfrac{y+6}{y^2-36}$ All but #2 have limits by substituting. #2 needs a different method.